Interaction between Unstable Optical Resonator and CW Chemical Laser

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Interaction between an unstable resonator and a cw diffusion-type chemical laser with turbulent or laminar mixing is analyzed by means of geometric optics and a simplified chemical model. Both cylindrical ($j = 1$) and spherical ($j = 2$) optics are considered. The gain region is assumed to be adjacent to one of the resonator mirrors. Laser performance is found to depend upon the parameters M, K_1, ζ_D, ζ_C, ζ_e, a, and γ, where M is the magnification, K_1 the ratio of pumping to collisional deactivation rate, ζ_D the normalized diffusion distance, ζ_C the normalized axial location of resonator axis, ζ_e the normalized axial length of resonator, a, the mirror absorptivity, and γ is related to the maximum possible zero power gain. Scaling laws are deduced and analytic solutions are obtained for the limiting cases $(M-1) \ll 1$ and $M^{-1} \ll 1$. In the former case, the laser medium is almost completely saturated and resonator efficiency is high. In the latter case, the radiation incident on the gain region originates near the optical axis and is uniform. It is concluded that optimum resonator efficiency is obtained when ζ_C is at the axial station where power-on gain is a maximum (other parameters being held constant). Nonsymmetric resonators $\zeta_C/(\zeta_e-\zeta_C) \neq 1$, in general, are required to achieve maximum output power. However, resonator efficiency is relatively insensitive to ζ_C location when the degree of optical saturation is high, and symmetric resonators can yield near optimum power in these cases.

Nomenclature

a = mirror absorptivity
B = σ/ε_p
$[F]_0$ = initial atomic fluorine concentration (mole/cc)
G = gain per semichannel [Eq. (3)]
G_o = zero-order solution for G [Eq. (19a)]
$\bar{G}$ = normalized gain per semichannel $[G/(\sigma w[F]_0)]$
g = local gain, $\sigma([HF_u]-[HF_1])$ (cm^{-1})
I = net local intensity $I^+ + I^-$ (w/cm^2)
I^+, I^- = intensity in $+y$ and $-y$ directions, respectively
I_2^+, I_2^- = incident and reflected intensity, respectively, at mirror 2 (Fig. 3)
I_{av} = average intensity in gain medium [Eq. (8c)]
J = BI_2^+/k_{c_d}
j = 1 or 2 for cylindrical or spherical mirrors, respectively
K_1 = k_f/k_{c_d}
k_f, k_{c_d} = forward pumping rate, collisional deactivation rate (s^{-1})
M = mirror magnification
n_{SC} = number of semichannels
P = output power per unit cavity height (w/cm)
$\bar{P}$ = normalized output power $2P/(\varepsilon_p u w n_{SC}[F]_0)$
$\bar{P}_u, \bar{P}_d$ = upstream and downstream output power, respectively
R_1, R_2 = mirror reflection coefficients [Eqs. (8)]
u = flow velocity (cm/sec)
w = semichannel width (cm)
x = streamwise distance from nozzle exit (Fig. 1)
x_C = cavity centerline location
x_D = diffusion distance, i.e., $y_f(x_D) = w$
y = distance parallel to cavity axis (Figs. 1 and 2)
α = $1+(2BI_{av}/k_{c_d})$
γ = maximum possible net gain $\sigma w[F]_0 n_{SC}$
ε = perturbation parameter $M-1$ or M^{-1}
ε_p = energy per mole of photons (J/mole)
ζ = xk_{c_d}/u
ζ^*, ζ^{**} = ray intercept points (Fig. 4)
ζ_C = cavity centerline location $x_C k_{c_d}/u$
ζ_e = downstream end of cavity $x_e k_{c_d}/u$
ζ_s = downstream end of cavity for saturated laser with maximum possible output $x_s k_{c_d}/u$
η_r = resonator efficiency P/P_s
σ = stimulated emission cross-section (cm^2/mole)
ϕ = Eqs. (6) and (A1)

Subscripts

0 = upstream of flame sheet or integration variable
0, 1 = zero order, first order, respectively
1, 2 = mirror 1, mirror 2, respectively
C = value at x_C
s = value for saturated laser

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I. Introduction

UNSTABLE optical resonators have been proposed by Siegman[1,2] for extracting energy from high-power lasers. A review of unstable optical resonator theory has been presented by Anan'ev.[3] Most of the theoretical studies of these resonators, to date, have been concerned with "empty cavities" or with a nonflowing gain medium. The effect of fluid flow is of critical importance, however, when analyzing the optical performance of cw chemical[4] or gasdynamic lasers.[5] Hence, a study of the interaction between an unstable resonator and a cw diffusion-type chemical laser flow medium was undertaken. The results are reported herein.

The primary objective of the present study is to determine the effect of the optical resonator magnification and centerline location on the near-field output power from a cw chemical laser. Geometric optics are used. The flowing gain medium is assumed to be adjacent to one of the resonator mirrors. Analytic solutions are obtained in the limits $M \to 1$ and $M \gg 1$, where M is cavity magnification. Effects of resonator and laser medium parameters on laser output power are then discussed. Turbulent diffusion is considered in the body of the report, and laminar diffusion is considered in the Appendix.

II. Analysis

A cw diffusion-type HF chemical laser that employs an unstable optical resonator is shown in Fig. 1. Multiple nozzles

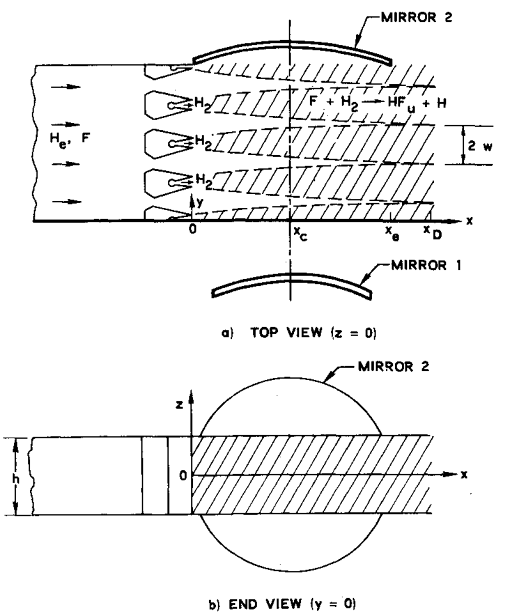


Fig. 1 Diffusion-type chemical laser with unstable resonator.

are used to expand an He–F mixture to a series of supersonic jets. Smaller scale nozzles expand H_2 into jets that diffuse laterally into the He–F streams. The reaction between H_2 and F [shaded region in Fig. 1a] creates vibrationally excited HF that lases in the optical cavity. The present coordinate system is also shown in Fig. 1. The centerline of the optical resonator is at $x = x_C$ in the $z = 0$ plane. For cylindrical optics ($j = 1$), the present solution will apply for $(-h/2) \leq z \leq (h/2)$, where h is nozzle height, while for spherical optics ($j = 2$), the present solution will apply for $z = 0$. In the following sections, a simplified model of the laser medium is discussed, a geometric model of the optical resonator is described, and analytic solutions for laser output power are given.

A. Simplified Model of Lasing Medium

The simplified diffusion laser model described in Ref. 6 is used. A single semichannel of the reactive flow is shown in Fig. 2. The reactants are assumed to be premixed but do not start to react until they reach a flame sheet location defined by $y_f = y_f(x)$ for $x \leq x_D$ and $y_f = w$ for $x > x_D$. Here, w is the nozzle exit semiwidth, and x_D is the diffusion distance and denotes the axial station where the flame sheet reaches the centerline of the He–F flow. Lateral diffusion of the reactants and products across streamtubes is neglected [the main effect of diffusion being accounted for by the choice for $y_f(x)$].

A two-level vibrational model is adopted. The local gain is $g = \sigma([HF_u]-[HF_1])$ (cm^{-1}), where σ is the cross section for stimulated emission or absorption (cm^2/mole), while $[HF_u]$ and $[HF_1]$ are the concentrations (mole/cm^3) of HF in the upper

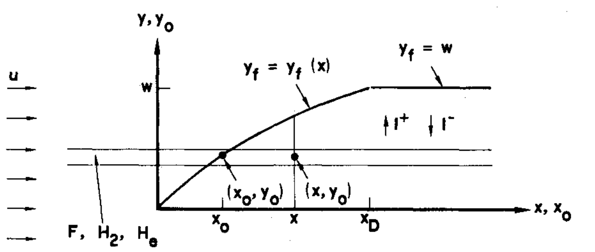


Fig. 2 Simplified model of mixing region. (Single semichannel is shown.)

and lower vibrational levels, respectively. The differential equation for the distribution of HF_u along each streamtube can be expressed by

$$u\frac{d[HF_u]}{dx} = k_f[F]-k_{c_d}[HF_u]-BI([HF_u]-[HF_1]) \qquad (1)$$

where $I = I^+ + I^-$ is the net local lasing intensity (w/cm^2), $B \equiv \sigma/\varepsilon_p$, and ε_p is the energy per mole of photons (J/mole). The first term on the right-hand side of Eq. (1) represents the creation of HF_u by the pumping reaction

$$[F]+[H_2] \overset{k_f}{\rightarrow} [HF_u]+[H],$$

while the second and third terms represent deactivation by collisions

$$[HF_u]+[M] \overset{k_{c_d}}{\rightarrow} [HF_1]+[M]$$

and by stimulated emission, respectively. The rates k_f and k_{c_d} are in units of s^{-1}, and mean values are used. Consideration of the mass conservation law $[F]_0 = [F]+[HF_u]+[HF_1]$, where $[F]_0$ is the initial concentration of F, and the assumption that k_f is constant permits Eq. (1) to be written in the form

$$\frac{d}{d\zeta}\left(\frac{g}{\sigma[F]_0}\right)+\left(1+\frac{2BI}{k_{c_d}}\right)\left(\frac{g}{\sigma[F]_0}\right)+1 = (1+K_1)(e^{-K_1(\zeta-\zeta_0)}) \qquad (2a)$$

where

$$\zeta = xk_{c_d}/u; \quad \zeta_0 = x_0k_{c_d}/u; \quad K_1 = k_f/k_{c_d} \qquad (2b)$$

Equation (2a) defines the variation of gain with x along a streamtube with ordinate y_0 that intersects the flame sheet at (x_0, y_0). The quantity ζ is the ratio of streamwise distance x to the characteristic collisional deactivation distance u/k_{c_d} and generally is of order 1. Henceforth, ζ will be used instead of x to denote streamwise distance. The quantity K_1 is the ratio of the pumping rate to the collisional deactivation rate and is generally large.

The net gain per semichannel is

$$G(\zeta) = \int_0^{y_f(\zeta)} g(\zeta, y_0)\,dy_0 \qquad (3)$$

An expression for G can be found by the integration of Eq. (2a) with respect to y_0 between the limits $0 \leq y_0 \leq y_f(\zeta)$. The variation of I with y is neglected, and an average value I_{av} is used,† where I_{av} is a function of ζ. Integration of Eq. (2a) then yields

$$w\left(\frac{d\bar{G}}{d\zeta}+\alpha\bar{G}\right) = (1+K_1)\int_0^{y_f} e^{-K_1(\zeta-\zeta_0)}\,dy_0 - y_f \qquad (4a)$$

where

$$\bar{G} = G/(\sigma w[F]_0)$$
$$\alpha = 1+(2BI_{av}/k_{c_d}) \qquad (4b)$$

The quantity $\bar{G}$ is the ratio of G to the maximum possible value of G. The quantity $\alpha-1$ is a measure of optical saturation.[6]

To simplify the development, the flame sheet is assumed to have the form

$$y_f = Ax = (w/\zeta_D)\zeta \qquad (\zeta \leq \zeta_D) \qquad (5)$$

where A is a constant of order 0.1. Equation (5) corresponds to turbulent mixing.‡ It follows that $y_0 = (w/\zeta_D)\zeta_0$ and $dy_0 = (w/\zeta_D)\,d\zeta_0$. Equation (4a) becomes

$$d\bar{G}/d\zeta + \alpha\bar{G} = \phi \qquad (6)$$

where

$$\zeta_D\phi = (1+K_1{}^{-1})(1-e^{-K_1\zeta})-\zeta \qquad (\zeta \leq \zeta_D)$$
$$= (1+K_1{}^{-1})[e^{-K_1\zeta}(e^{K_1\zeta_D}-1)]-\zeta_D \qquad (\zeta > \zeta_D)$$

† The local lasing intensity I determines the rate at which HF_u is deactivated by stimulated emission. Use of an average value of I is consistent with the use of a mean value of k_{c_d}.

‡ More generally, $y_f \sim x^N$, where $N = 1$ or $\frac{1}{2}$ for turbulent or laminar mixing, respectively.

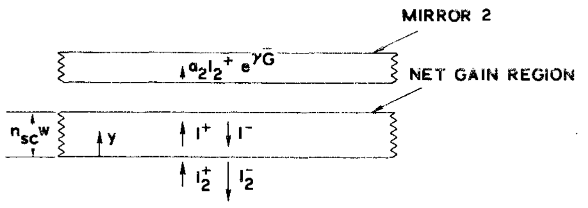


Fig. 3 Amplification of incident radiation by gain region adjacent to mirror 2.

The initial condition is $\bar{G}(0) = 0$. When α is constant, integration of Eq. (6) for $\zeta \leqq \zeta_D$ yields

$$\zeta_D \bar{G} = (1+K_1{}^{-1})\left[\frac{1-e^{-\alpha\zeta}}{\alpha} - \frac{e^{-K_1\zeta}-e^{-\alpha\zeta}}{\alpha - K_1}\right] - \frac{\alpha\zeta - 1 + e^{-\alpha\zeta}}{\alpha^2} \quad (7)$$

In the following development, only the case $\zeta \leqq \zeta_D$ is considered.

Equation (6) defines the conditions along each of the semichannels in Fig. 1. The integration of Eq. (6) is difficult, in general, since α varies with ζ as well as with semichannel location, and the interaction with the optical resonator must be considered. A simplified treatment of the multiple semichannel gain region follows.

B. Reflection Coefficients

Expressions for the ratio of reflected to incident radiation at each mirror are obtained. The gain region is assumed to be adjacent to mirror 2 (Fig. 3). n_{SC} is the number of semichannels. The net width of the gain region is wn_{SC}, which is assumed to be small compared with the distance between the mirrors. $I_2{}^+$ denotes the intensity of the radiation incident on the gain region. The reflected radiation, from the gain-mirror combination, is $I_2{}^-$. It is assumed that at each streamwise station the net gain per semichannel G is the same for each semichannel. The average value of the local gain at each streamwise station is then $g_{av} = G/w$, and the net gain at each station is

$$g_{av}\, wn_{SC} = Gn_{SC} = \gamma\bar{G}$$

where $\gamma = \sigma w[\mathrm{F}]_0 n_{SC}$ is a measure of the maximum possible net gain. It follows that I^+ and I^- increase exponentially in the $+y$ and $-y$ directions, respectively. Thus, $I^+/I_2{}^+ = e^{g_{av}y}$ and $I^-/I_2{}^- = e^{-g_{av}y}$. The ratio of reflected to incident radiation is then

$$R_2 \equiv I_2{}^-/I_2{}^+ = (1-a_2)\,e^{2\gamma\bar{G}} \quad (8a)$$

where a_2 is the absorption coefficient at the surface of mirror 2.§ The corresponding expression for the reflection coefficient at mirror 1 is

$$R_1 = (1-a_1) \quad (8b)$$

The average value of net local intensity is defined by

$$I_{av} \equiv \frac{1}{wn_{SC}}\int_0^{wn_{SC}} (I^+ + I^-)\,dy$$

Integration yields

$$\gamma\bar{G}(I_{av}/I_2{}^+) = (1-a_2)\,e^{2\gamma\bar{G}} - 1 + a_2\,e^{\gamma\bar{G}} \quad (8c)$$

Equation (8c) provides the average value of intensity for use in Eq. (6). The present estimate for I_{av} conserves energy. This is seen by writing Eq. (8c) in the form

$$\gamma\bar{G}I_{av} = (I_2{}^- - I_2{}^+) + a_2 I_2{}^+\,e^{\gamma\bar{G}}$$

Here, the left-hand side is the net power (per unit mirror area) generated by stimulated emission. The first term on the right-hand side is the net power leaving the gain region, while the second term is the power absorbed by the mirror.

In the present method of solution for R_2, it is assumed that $\bar{G}$ is the same in each semichannel. This assumption is exact when $\gamma\bar{G} \ll 1$, regardless of the degree of saturation $\alpha - 1$. The assumption is also exact in the limits $\alpha = 1$ and $\alpha \to \infty$ for arbitrary values of $\gamma\bar{G}$. The solution for R_2, however, is approximate for intermediate values of α and $\gamma\bar{G}$ not small. The present approach is considerably simpler than alternative approaches in which the variation of $\bar{G}$ is considered and is consistent with the approximate nature of the present model.

When $\gamma\bar{G} \ll 1$, Eqs. (8a) and (8c) become

$$R_2 = (1-a_2)[1+2\gamma\bar{G}+O(\gamma\bar{G})^2] \quad (9a)$$

$$I_{av}/I_2{}^+ = (2-a_2)[1+O(\gamma\bar{G})] \quad (9b)$$

This "small gain" approximation is frequently used in studies of stable cavities[7] but cannot be used in studies of unstable resonators with large magnification since large values of $\gamma\bar{G}$ are needed to reach the threshold condition, as will be shown.

C. Geometric Model of Unstable Resonator

An unstable resonator with the gain region adjacent to mirror 2 is shown in Fig. 4. The upstream edge of mirror 2 is assumed to be located at the nozzle exit at $\zeta = 0$. The axis of the resonator is at ζ_C, and the downstream edge of mirror 2 is at ζ_e. The resonator need not be symmetric about ζ_C. Cylindrical ($j = 1$) and spherical ($j = 2$) mirrors are considered.

In the geometric model of the unstable resonator, the radiation reflected from each mirror appears to originate from a virtual center on the resonator axis.[1] The location of the virtual center for each mirror depends on the mirror radii and their separation. It follows that stimulated emission can be assumed to commence in the vicinity of the axis and then diverge outward. A typical radiation path is indicated in Fig. 4.

An expression is derived that relates the intensity of an arbitrary ray at two successive interceptions with mirror 2, i.e., at ζ^* and ζ in Fig. 4. The intercept with mirror 1 is denoted ζ^{**}. (For a confocal system, $\zeta^* = \zeta^{**}$, and I^- is parallel to the

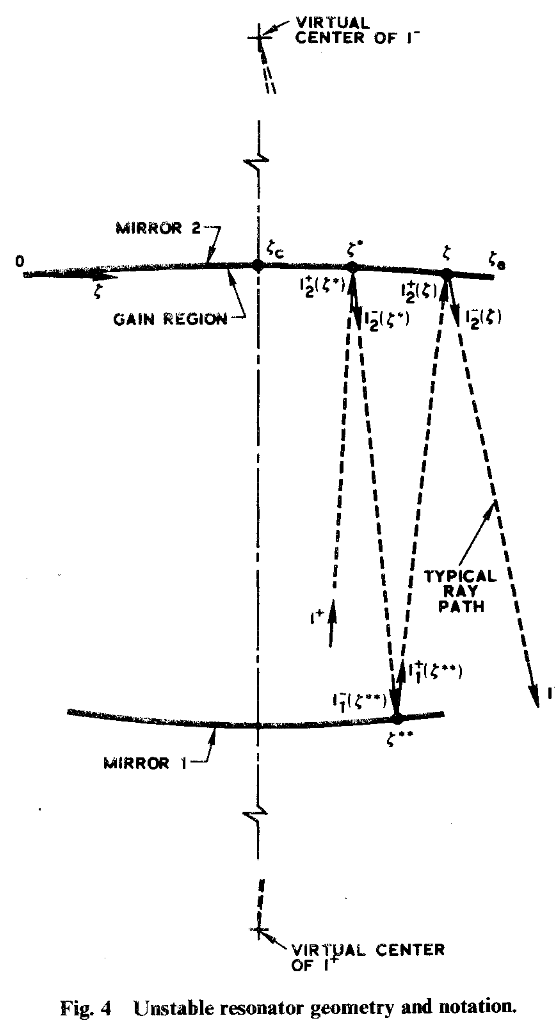


Fig. 4 Unstable resonator geometry and notation.

§ For partially transmitting mirrors, a_2 is replaced by $a_2 + t_2$, where t_2 is transmissivity.

resonator axis.) Note that $\zeta - \zeta_C$ denotes the distance of point ζ from the resonator axis. A property of an unstable resonator[1] is that

$$(\zeta - \zeta_C)/(\zeta^* - \zeta_C) = M \tag{10}$$

where M is the round-trip magnification, and its value also depends on the mirror radii and their separation. From conservation of energy, it can be shown that the intensity of the radiation at mirrors 1 and 2 is related by

$$I_1^-(\zeta^{**})/I_2^-(\zeta^*) = [(\zeta^* - \zeta_C)/(\zeta^{**} - \zeta_C)]^j \tag{11a}$$

$$I_2^+(\zeta)/I_1^+(\zeta^{**}) = [(\zeta^{**} - \zeta_C)/(\zeta - \zeta_C)]^j \tag{11b}$$

Assume that R_1 is constant.¶ It follows from Eqs. (9–11) that the intensity at ζ^* and ζ are related by

$$J(\zeta) = R_1 R_2(\zeta^*) J(\zeta^*)/M^j \tag{12a}$$

$$\zeta - \zeta_C = M(\zeta^* - \zeta_C) \tag{12b}$$

where $J \equiv BI_2^+/k_{c_d}$. Evaluation of Eq. (12a) at ζ_C and the use of Eqs. (8a) and (8b) indicate

$$2\gamma\bar{G}(\zeta_C) = \ln\{M^j/[(1-a_1)(1-a_2)]\} \tag{13}$$

Equation (13) provides an additional boundary condition on $\bar{G}$ (for later use) and expresses the fact that the net gain at ζ_C equals the net losses resulting from mirror absorption and beam divergence. As previously stated, large values of $\gamma\bar{G}$ are needed when M^j is large.

Equation (12) can be viewed as a parametric equation (with ζ^* as the parameter) relating J with ζ. The simultaneous solution of Eqs. (6) and (12) for the variation of J and $\bar{G}$ with ζ defines the performance of an unstable resonator. The solution is valid for all z when $j = 1$ and is valid for $z = 0$ when $j = 2$.

D. Output Power

Expressions for laser output power are deduced. P denotes the net output power per unit nozzle height. In the present two-level model, the maximum possible output power per unit height from the reactants is $\varepsilon_p uwn_{SC}[\mathrm{F}]_0/2$, which corresponds to one photon being emitted for every two molecules of HF_u formed by the pumping reaction. The latter power is obtained when the lasing medium is saturated and when mirror losses and collisional deactivation effects are negligible. Introduce $\bar{P}$ such that

$$\bar{P} \equiv P/[\varepsilon_p uwn_{SC}[\mathrm{F}]_0/2] = 2BP/\gamma u \tag{14}$$

Thus, $\bar{P}$ is the ratio of actual output power to the maximum possible power and is a measure of laser chemical efficiency.

Resonator output power can be found by either of two approaches. In an edge-coupled resonator, the radiation reflected from regions $0 \leq \zeta \leq [(M-1)/M]\zeta_C$ and $[\zeta_e + (M-1)\zeta_C]/M \leq \zeta \leq \zeta_e$ on mirror 2 is extracted from the cavity and constitutes the laser output. These limits are deduced by setting $\zeta = 0$, ζ_e in Eq. (12b), and solving for ζ^*. The net output power is then

$$\bar{P} = (2/\gamma)\int_0^{[(M-1)/M]\zeta_C} R_2 J\, d\zeta + (2/\gamma)\int_{[\zeta_e + (M-1)\zeta_C]/M}^{\zeta_e} R_2 J\, d\zeta$$

$$\equiv \bar{P}_u + \bar{P}_d \tag{15}$$

where $\bar{P}_u$ and $\bar{P}_d$ represent the power coupled out of the upstream and downstream edges of the resonator, respectively.

The net output power can also be found by evaluating the power generated in the gain region

$$P_G \equiv \gamma \int_0^{\zeta_e} \bar{G} I_{av}\, dx$$

and subtracting from this the power absorbed by the mirrors. Thus,

$$\bar{P}_G \equiv 2BP_G/\gamma u = \int_0^{\zeta_e} \bar{G}(\alpha - 1)\, d\zeta$$

$$= \int_0^{\zeta_e} \phi\, d\zeta - \int_0^{\zeta_e} \bar{G}\, d\zeta - \bar{G}(\zeta_e) \tag{16a}$$

¶ More generally, $R_1 = R_1(\zeta^{**})$ in (12a), and ζ^{**} is related to ζ by the magnification in going from mirror 1 to mirror 2.

where Eq. (6) and the boundary condition $\bar{G}(0) = 0$ have been used. $\bar{P}_G$ increases as the degree of saturation increase, i.e., as $\bar{G} \to 0$. From the definition of ϕ, for $\zeta \leq \zeta_D$,

$$\int_0^{\zeta_e} \phi\, d\zeta = \frac{1}{\zeta_D}\left[(1 + K_1^{-1})\left(\zeta_e + \frac{e^{-K_1\zeta_e} - 1}{K_1}\right) - \frac{\zeta_e^2}{2}\right] \tag{16b}$$

The power generated in the upstream and downstream portions of the resonator are found by evaluating Eq. (16a) between the limits 0 to ζ_C and ζ_C to ζ_e, respectively. The power absorbed on mirror 2 is

$$\bar{P}_{a_2} = (2a_2/\gamma)\int_0^{\zeta_e} J\, e^{\gamma\bar{G}}\, d\zeta \tag{17}$$

A similar expression can be deduced for mirror 1.

When the medium is optically saturated, $\bar{G} \to 0$, $I_{av} \to \infty$, and $\bar{G}I_{av}$ remains finite. The length of the lasing region, in this limit, is ζ_s and is found by setting $\phi = 0$. The result is

$$\zeta_s = (1 + K_1^{-1})(1 - e^{-K_1\zeta_s}) \tag{18a}$$

The corresponding net output power, from the gain medium, is found by setting $\bar{G} = 0$ in Eq. (16a) and equals

$$\zeta_D \bar{P}_s = \zeta_s - (\zeta_s^2/2) \tag{18b}$$

This is the maximum possible power that can be extracted from the lasing medium. Values of ζ_s and $\zeta_D \bar{P}_s$ are given in Table 1 as a function of K_1.

A "resonator efficiency" can be defined such that

$$\eta_r = \bar{P}/\bar{P}_s \tag{18c}$$

The equantity η_r is the ratio of the actual laser output power to the output power, which is obtained when the medium is saturated.

E. Scaling Laws

Unstable resonator performance is obtained by the simultaneous solution of Eqs. (6) and (12) with the aid of Eqs. (8) and (13). Numerical solutions are required in general. However, analytic solutions can be obtained for special cases. Discussion of some of these is given later.

This system of equations also defines the scaling laws and parameters for the unstable resonator-chemical laser combination. The dependent variables $\gamma\bar{G}$ and J are functions of the independent variable ζ, and the parameters a_1, a_2, K_1, M, ζ_C, ζ_D, and γ. For $\zeta \leq \zeta_D$, ζ_D and γ appear only in the form γ/ζ_D. When $\zeta_e \leq \zeta_D$, the quantity γ/ζ_D is a measure of the maximum possible net zero power gain including diffusion effects. $\gamma\bar{G} \equiv Gn_{SC}$ is the net gain at each axial station. The output power $\gamma\bar{P}$ is a function of ζ_e as well as the above parameters. For $\zeta_e \leq \zeta_D$, the saturated laser output power $\zeta_D\bar{P}_s$ is a function only of K_1 (Table 1).

The resonator can be shown to be saturated when, for $\zeta_e \leq \zeta_D$,

$$(\zeta_D/\gamma)\ln[M^j(1-a_1)^{-1}(1-a_2)^{-1}] \ll 1$$

The left-hand side is, roughly, the ratio of power-on gain, from Eq. (13), to zero-power gain. The latter, from Eq. (6), is $\bar{G} = 0(\zeta_D^{-1})$. The previous expressions apply for turbulent mixing. For laminar mixing, ζ_D is replaced by $\zeta_D^{1/2}$ (Appendix A).

F. Analytic Solutions

Analytic solutions have been obtained in the limits of magnification near one $M - 1 \equiv \varepsilon \ll 1$ and large magnification $M^{-1} \equiv \varepsilon \ll 1$. Discussions of these solutions follow. It is assumed that $\zeta_e \leq \zeta_D$.

Table 1 Output power and lasing length for saturated laser with $\zeta_D \geq \zeta_s$[a]

K_1	10^6	10^2	10	5	2.5	1.0
ζ_s	1.000	1.010	1.100	1.197	1.352	1.594
$\zeta_D\bar{P}_s$	0.500	0.500	0.495	0.481	0.438	0.324

[a] From Eq. (18).

1) *Case* $M-1 \equiv \varepsilon \ll 1$

The dependent variables are expressed in the forms

$$\bar{G} = \varepsilon G_0[1+0(\varepsilon)] \tag{19a}$$

$$J = (J_0/\varepsilon)[1+0(\varepsilon)] \tag{19b}$$

where G_0 and J_0 are functions of ζ. The superscript bar has been omitted from G_0. From Eqs. (3) and (13), the boundary conditions on G_0 are

$$G_0(0) = 0 \tag{20a}$$

$$G_0(\zeta_C) = j/2\gamma \tag{20b}$$

Substitution into Eq. (6) yields

$$J_0 G_0 = (\phi/4)[1+0(\varepsilon)] \tag{21}$$

For convenience, the notation for points ζ and ζ^* are reversed in Fig. 4, and Eq. (12) becomes

$$J(\zeta^*) = R_2(\zeta)J(\zeta)/M^j \tag{22a}$$

$$\zeta^* - \zeta = \varepsilon(\zeta - \zeta_C) \tag{22b}$$

$\zeta^* - \zeta$ is of order ε. Thus, the axial separation of successive intercepts of a ray with mirror 2 is small. $J(\zeta^*)$ can be expressed in terms of $J(\zeta)$, and its derivatives by a Taylor expansion about ζ. The result, from Eq. (22b), is

$$J(\zeta^*) = J(\zeta) + J'(\zeta)(\zeta-\zeta_C)\varepsilon + 0(\varepsilon^2) \tag{23}$$

where the primes denote differentiation with respect to ζ. Substitution into Eq. (22a) yields

$$J_0'(\zeta-\zeta_C) + jJ_0 = 2\gamma G_0 J_0[1+0(\varepsilon)+0(a/\varepsilon)] \tag{24}$$

where it is assumed that $a = 0(\varepsilon^2)$. Equations (21) and (24) along with the boundary conditions in Eq. (20) define the zero-order solution. Substitution of Eqs. (21) and (24) and integration between the limits ζ and ζ_C yield

$$\frac{2\zeta_D}{\gamma}J_0 = \frac{1+K_1^{-1}-\zeta_C}{j} - \frac{\zeta-\zeta_C}{j+1} + \frac{e^{-K_1\zeta_C}(1+K_1^{-1})}{[K_1(\zeta-\zeta_C)]^j}\{[e^{-K_1(\zeta-\zeta_C)}][K_1(\zeta-\zeta_C)+1]^{j-1}-1\} \tag{25a}$$

$$2\gamma G_0 = [(1+K_1^{-1})(1-e^{-K_1\zeta}) - \zeta]/[2\zeta_D J_0/\gamma] \tag{25b}$$

which satisfy the boundary conditions on $G_0(0)$ and $G_0(\zeta_C)$.

G_0 and J_0 can be used to find the output power to first order. For $a_1 = a_2 = 0(\varepsilon^2)$, the output power can be expressed

$$\bar{P} = \bar{P}_0\{1-\varepsilon[\bar{\bar{P}}_1 + (a/\varepsilon^2)\bar{\bar{P}}_a]\} \tag{26}$$

where $\bar{P}_0$ is the zero-order output power, $\bar{\bar{P}}_1$ the decrement in output power due to decreased saturation as M departs from 1, and $\bar{\bar{P}}_a$ is the power lost by mirror absorption. In the present limit, both mirrors have the same absorption, and the net absorption equals twice the value indicated by Eq. (17). From Eq. (16) and (17),

$$\zeta_D(\bar{P}_u)_0 = \left[(1+K_1^{-1})\left(\zeta + \frac{e^{-K_1\zeta}}{K_1}\right) - \frac{\zeta^2}{2}\right]_{\zeta=0}^{\zeta=\zeta_C} \tag{27a}$$

$$\zeta_D(\bar{P}_d)_0 = \{\ \}_{\zeta=\zeta_C}^{\zeta=\zeta_e} \tag{27b}$$

$$\zeta_D\bar{P}_0 = \{\ \}_{\zeta=0}^{\zeta=\zeta_e} \tag{27c}$$

$$\bar{\bar{P}}_1\bar{P}_0 = \int_0^{\zeta_e} G_0\,d\zeta + G_0(\zeta_e) \tag{27d}$$

$$\bar{\bar{P}}_a\bar{P}_0 = (4/\gamma)\int_0^{\zeta_e} J_0\,d\zeta \tag{27e}$$

The net zero-order power $\bar{P}_0$ is independent of j and ζ_C, and, for $\zeta_e = \zeta_s$, equals the saturated output power $\bar{P}_s$ given by Eq. (18b). The optical efficiency of the resonator is obtained from

$$(1-\eta_r)/\varepsilon = \bar{\bar{P}}_1 + (a/\varepsilon^2)\bar{\bar{P}}_a \tag{27f}$$

These results can be illustrated by the consideration of $K_1 \to \infty$, $\zeta_e = \zeta_s = 1$, and $0 < \zeta_C < 1$. It is found that

$$(2\zeta_D/\gamma)J_0 = [(j+1) - \zeta_C - j\zeta]/[j(j+1)] \tag{28a}$$

$$2\gamma G_0 = [j(j+1)(1-\zeta)]/[(j+1) - \zeta_C\zeta - j\zeta] \quad (\zeta \neq 0) \tag{28b}$$

$$2\gamma G_0 = 0 \quad (\zeta = 0) \tag{28c}$$

$$\zeta_D(\bar{P}_u)_0 = \zeta_C[1-(\zeta_C/2)] \tag{28d}$$

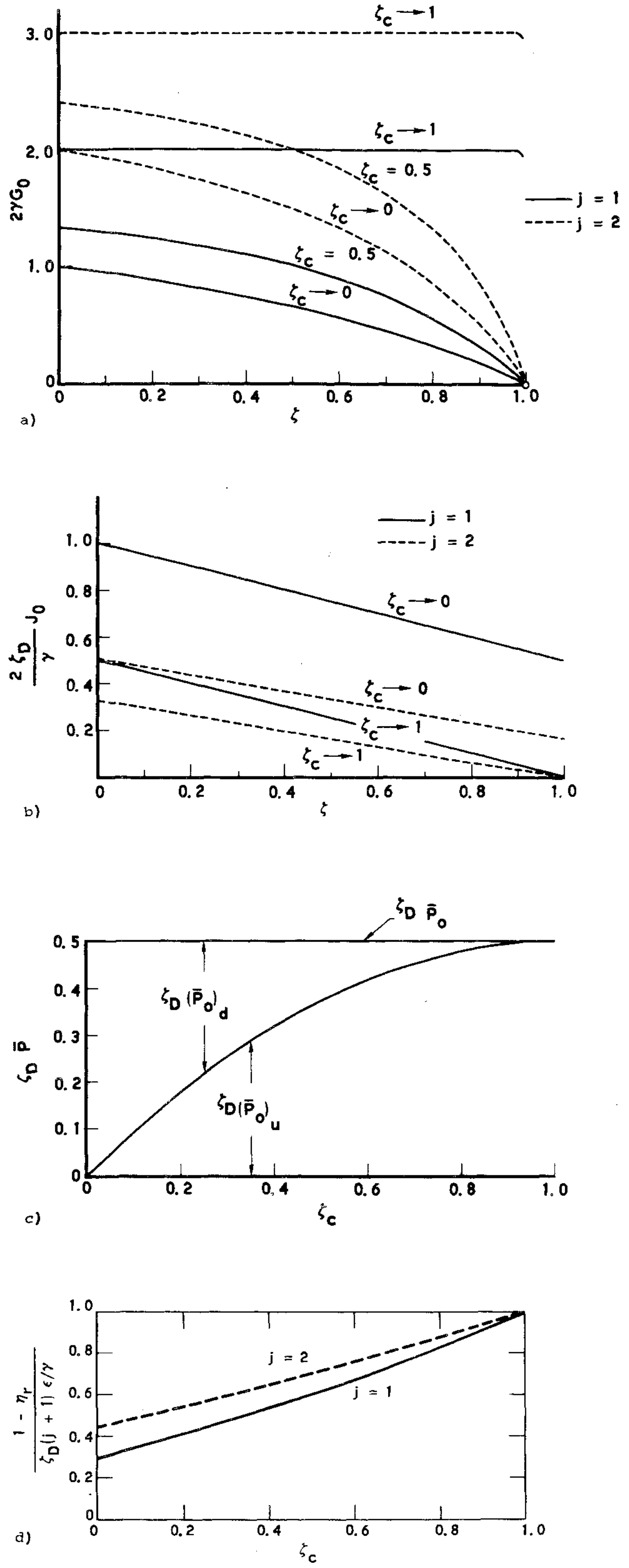


Fig. 5 Unstable resonator performance in limit $M-1 \ll 1$, $K_1 \to \infty$, $0 < \zeta_C < 1$, and $\zeta_D \geq \zeta_e = 1$ (Eq. 28). a) Zero-order gain in interval $0 < \zeta < 1$. ($G_0(0) = 0$); b) Zero-order incident and reflected intensity in interval $0 \leq \zeta \leq 1$; c) Zero-order output; d) Resonator efficiency decrement in absence of mirror absorption [$a = 0(\varepsilon)^3$].

$$\zeta_D(P_d)_0 = (1/2) - \zeta_C[1-(\zeta_C/2)] \quad (28e)$$

$$\zeta_D \bar{\bar{P}}_0 = 1/2 \quad (28f)$$

$$\frac{\gamma}{\zeta_D}\frac{\bar{\bar{P}}_1}{j+1} = 1 - \frac{(1-\zeta_C)}{j}\ln\left(\frac{1+j-\zeta_C}{1-\zeta_C}\right) \quad (28g)$$

$$\bar{\bar{P}}_a = 2(j+2-2\zeta_C)/[j(j+1)] \quad (28h)$$

Equation (28) is plotted in Fig. 5. The gain G_0 approaches a maximum as $\zeta \to 0$ for all $0 < \zeta_C < 1$ (Fig. 5a). The resonator becomes more saturated as $\zeta_C \to 0$. Hence, G_0 decreases and J increases (Fig. 5b) as $\zeta_C \to 0$. The ratio of upstream to downstream power extraction also decreases as $\zeta_C \to 0$ (Fig. 5c). With mirror absorption neglected, laser output power increases with decrease in ε, ζ_C, and j (Fig. 5d). However, mirror absorption losses also increase with a decrease in ε. Hence, for a given value of mirror absorptivity, there is a corresponding value of ε that yields a maximum output power. When $K_1 \to \infty$, the optimum value of ε is, from Eqs. (27e), (28g), and (28h),

$$\frac{\zeta_D}{\gamma}\frac{\varepsilon^2}{a} = \frac{2}{(j+1)^2}\frac{j+2(1-\zeta_C)}{j-(1-\zeta_C)\ln[(1+j-\zeta_C)/(1-\zeta_C)]} \quad (29)$$

Since it has been assumed that $a/\varepsilon^2 = 0(1)$, it follows that Eq. (29) is applicable when $\zeta_D/\gamma = 0(1)$.

2) *Case $M^{-1} \equiv \varepsilon \ll 1$*

ζ^* and ζ are successive interceptions of a ray with mirror 2 as indicated in Fig. 4. From Eq. (12), conditions at ζ^* and ζ are related by

$$J(\zeta) = \varepsilon^j R_2(\zeta^*)J(\zeta^*) \quad (30a)$$

$$\zeta - \zeta_C = \varepsilon^{-1}(\zeta^* - \zeta) \quad (30b)$$

$\zeta^* - \zeta_C$ is of order ε for $\zeta - \zeta_C$ of order one. Thus, all rays striking mirror 2 originate in the vicinity of the optical axis ζ_C, as expected physically, for large M. Expansion of $R_2(\zeta^*)$ and $J(\zeta^*)$ in a Taylor series about ζ_C, substitution in Eq. (30), and collection of terms of order ε^0 yield

$$J(\zeta) = J_C[1+0(\varepsilon)+0(a_1)] \quad (31)$$

where $J_C \equiv J(\zeta_C)$ and $a_1 \leq 0(\varepsilon)$ is assumed. Thus, the incident radiation on mirror 2 is uniform to zero order and can be shown to be a linear function of ζ to first order. The evaluation of the constant on the right-hand side of Eq. (31) requires consideration of Eq. (6). The latter becomes

$$d\bar{G}/d\zeta + \bar{G} + (2J_C/\gamma)\,e^{2\gamma\bar{G}}[1+0(\varepsilon)] = \phi \quad (32)$$

Analytic solutions can be found for special cases. The zero output power solution, i.e., $J_C = 0$, is given by Eq. (7) with $\alpha = 1$. For a saturated medium, i.e., $(\zeta_D/\gamma)\ln M^j \ll 1$, the first two terms on the left-hand side of Eq. (32) are negligible. Equations (32) and (13) then yield

$$2\gamma\bar{G} = \ln\{(M^j\phi/\phi_C)[1+0(\varepsilon)+0(\zeta_D\gamma^{-1}\ln M^j]\} \quad (33a)$$

$$J = [\gamma\phi_C/(2M^j)][1+0(\varepsilon)+0(\zeta_D\gamma^{-1}\ln M^j)] \quad (33b)$$

where $\phi_C = \phi(\zeta_C)$. The corresponding output power is obtained from Eq. (16a) with $\bar{G} = 0$. The ratio of power absorbed by the mirror to laser output power is of order $a_2\varepsilon^{j/2}$.

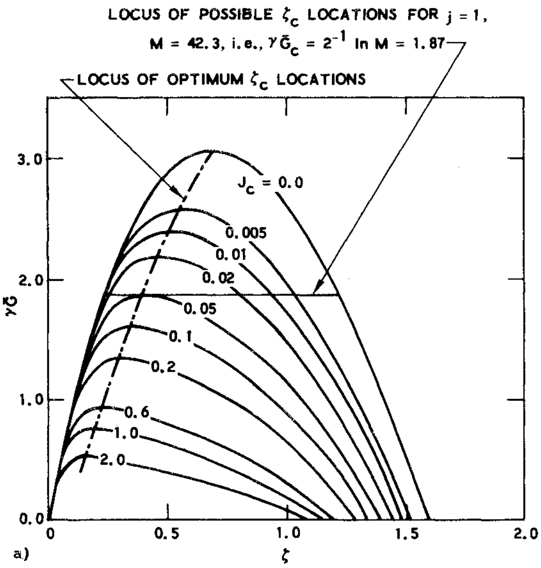
Equation (32) has been integrated for $K_1 \to \infty$, $\gamma/\zeta_D = 10$, and various values of J_C to illustrate the $M^{-1} \ll 1$ results. The variation of gain and of reflected intensity with ζ is given in Fig. 6. Permissible values of ζ_C, for a given M, can be deduced from Fig. 6a. The procedure is shown in Fig. 6a for case $j = 1$, $M = 42.3$, i.e., $\gamma\bar{G}_C = 2^{-1}\ln M = 1.87$. For this case, lasing occurs for all $0.24 < \zeta_C < 1.23$. For each ζ_C, the corresponding value of J_C can be deduced from Fig. 6, and the corresponding optimum cavity length ζ_e, i.e., $\bar{G}(\zeta_e) = 0$, and resonator efficiency can be found from Table 2. The resonator efficiency is a maximum when J_C has its maximum possible value. This occurs when ζ_C is located such that the power-on gain curve has a maximum at ζ_C, i.e., $\bar{G}'(\zeta_C) = 0$. For $M = 42.3$; $K_1 \to \infty$, $\gamma/\zeta_D = 10$, the optimum cavity centerline location is at $\zeta_C = 0.40$, and the corresponding value of J_C is 0.05. The optimum cavity length is $\zeta_e = 1.39$, and the resonator efficiency is 0.50 (Table 2). The optimum cavity for $M = 42.3$ is nonsymmetrical since $\zeta_e \neq 2\zeta_C$. When the degree of saturation is high, the output power is less sensitive to the location of ζ_C and a symmetric resonator with $\zeta_e = \zeta_s$ and $\zeta_C = \zeta_s/2$ can be used. The present conclusions regarding optimum ζ_C location agree with those deduced for $M - 1 \ll 1$, with mirror absorption neglected, and appear to apply for all M.

Table 2 Optimum resonator length and resonator efficiency as function of J_C for case $M^{-1} \ll 1$, $K_1 \to \infty$, $\gamma/\zeta_D = 10$, and $\zeta_e \leq \zeta_D$ [Eqs. (32), (16), and (18)]

J_C	0	0.005	0.01	0.02	0.05	0.10	0.20	0.60	1.00	2.00
ζ_e	1.59	1.52	1.49	1.45	1.39	1.34	1.29	1.19	1.15	1.10
η_r	0	0.21	0.29	0.38	0.50	0.59	0.69	0.82	0.87	0.92

III. Conclusions

The geometric optics approximation used here is valid for resonators with large equivalent Fresnel numbers.[8] For a confocal unstable resonator, the equivalent Fresnel number is $N_{EQ} = (w_2^2/\lambda L)[(M-1)/M]$, where w_2 is the half-width of the larger mirror, λ is the wavelength, and L is the separation



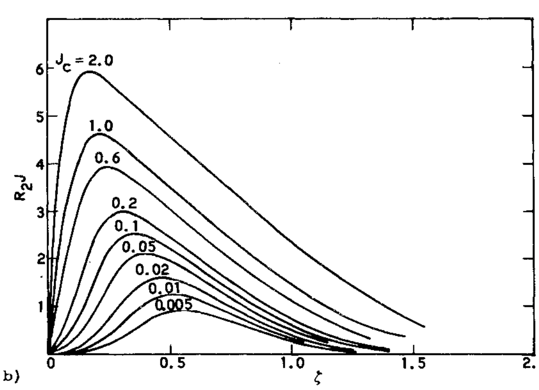


Fig. 6 Variation of gain and reflected intensity with ζ for case $M^{-1} \ll 1$, $K_1 \to \infty$, $\gamma/\zeta_D = 10$, $a = 0(\varepsilon)$, and $\zeta_D \geq \zeta_e$ [Eq. (32)]: a) Gain distribution; b) Reflected intensity $R_2J \equiv BI_2^-/k_{c_d}$.

distance between the mirrors. For typical kilowatt-level HF(DF) chemical lasers,[4,9] $w_2 = 4$, $\lambda = 3\times10^{-4}$, $L = 10^2$ cm, and $N_{EQ} \doteq 500(M-1)/M$. Hence, N_{EQ} is large for all M except M very close to one, and the geometric optics model is generally valid.

A numerical computer code describing the performance of a chemical laser-unstable resonator combination has been developed by Bullock.[10] In this code, physical optics, multiple vibrational levels, and more complex chemical kinetics than used here are used. The small-gain approximation Eq. (9) is used, but this can be easily modified. Numerical codes of this type have the potential of providing accurate estimates of chemical laser performance. The merit of the present approach is that the basic parameters are identified, the dependence of resonator performance on these parameters is established, and analytic solutions can be obtained for special cases. A comparison with Ref. 10 is beyond the scope of the present study.

Appendix: Laminar Mixing

The effect of laminar mixing on unstable resonator performance is briefly discussed herein for the case $\zeta_D \geqq \zeta_e$. The flame-sheet location is of the form $y_f \sim x^{1/2}$. The streamwise distribution of gain is found from Eq. (6) with

$$(K_1\zeta_D)^{1/2}\phi = (1+K_1)D[(K_1\zeta)^{1/2}] - (K_1\zeta)^{1/2} \qquad \text{(A1)}$$

where

$$D(Z) \equiv e^{-Z^2}\int_0^Z e^{Z_0^2}\,dZ_0$$

is the Dawson integral. Analytical solutions have been obtained for the limiting case $M \to 1$ and $M \to \infty$.

A. Case $M-1 \equiv \varepsilon \ll 1$

Equations (21) and (24) are unchanged except for the redefinition of ϕ. The medium is saturated, to zero order, and the extent of the lasing zone is found from $\phi(\zeta_s) = 0$, or

$$(K_1\zeta_s)^{1/2} = (1+K_1)D[(K_1\zeta_s)^{1/2}] \qquad \text{(A2)}$$

The limit $K_1 \to \infty$ is obtained by noting $D[(K_1\zeta)^{1/2}] \to [2(K_1\zeta)^{1/2}]^{-1}$ as $K_1 \to \infty$, $\zeta \neq 0$. In this limit, $\zeta_s = 1/2$. The zero-order output power for $\zeta_e = \zeta_s$ is

$$\zeta_D^{1/2}\overline{P}_0 = \zeta_D^{1/2}\overline{P}_S = [(1+K_1)/K_1^{1/2}]D[K_1\zeta_s)^{1/2}] - (2/3)\zeta_s^{3/2} \qquad \text{(A3a)}$$

$$\zeta_D^{1/2}\overline{P}_0 = 2^{1/2}/3 \qquad (K_1 \to \infty) \qquad \text{(A3b)}$$

When $j = 1$, the result for J_0 is

$$J_0 = \frac{\gamma}{2\zeta_D^{1/2}}\frac{1}{\zeta-\zeta_C}\left\{\frac{1+K_1^{-1}}{K_1^{1/2}}[\![(K_1\zeta)^{1/2} - D[(K_1\zeta)^{1/2}]]\!] - \frac{2}{3}\zeta^{3/2}\right\}_{\zeta=\zeta_C}^{\zeta=\zeta} \qquad \text{(A4)}$$

and $G_0 = \phi/(4J_0)$

B. Case $M^{-1} \equiv \varepsilon \ll 1$

The zero-power solution is

$$\zeta_D^{1/2}\overline{G} = \frac{1+K_1}{K_1^{1/2}}\frac{D[(K_1\zeta)^{1/2}]}{1-K_1} - \frac{2K_1}{1-K_1}D[\zeta^{1/2}] - \frac{\zeta^{1/2}}{2} \qquad \text{(A5)}$$

The medium is saturated when $\zeta_D^{1/2}\gamma^{-1}\ln M^j \ll 1$. Equation (33) applies except for the redefinition of ϕ.

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